

# Quantization in a General Light-front Frame

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In this paper, we study the question of quantization of quantum field theories in a general light-front frame. We quantize scalar, fermion as well as gauge field theories in a systematic manner carrying out the Hamiltonian analysis carefully. The decomposition of the fields into positive and negative frequency terms needs to be done carefully after which we show that the (anti) commutation relations for the quantum operators become frame independent. The frame dependence is completely contained in the functions multiplying these operators in the field decomposition. We derive the propagators from the vacuum expectation values of the time ordered products of the fields.

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## I. INTRODUCTION

Light-front field theories [1] have been studied vigorously in the past. The conventional light-front frame where one defines

$$\bar{x}^\mu = (x^+, x^i, x^-), \quad x^\pm = x^0 \pm x^3, \quad i = 1, 2, \quad (1)$$

also describes the infinite momentum frame where many physical results simplify considerably [2]. The quantization of quantum field theories on the light-front in the conventional light-front frame (1) has been studied in detail in [3, 4, 5, 6]. One of the many advantages of using a light-front quantization is that a larger number of generators of the Poincare algebra become kinematical [1] leading to a trivial structure for the vacuum state [7]. This, in principle, allows for the possibility of carrying out nonperturbative studies in a simple manner.

More recently, it has been observed [8, 9, 10] that description of statistical mechanics for field theories quantized on the light-front prefers an oblique coordinate frame

$$\bar{x}^\mu = (x^+, x^i, x^3), \quad (2)$$

with  $x^+$  defined in (1). The advantage of this frame lies in the fact that the associated temperature can be identified with that for the theory in Minkowski space quantized on an equal time surface. In general, a frame defined by

$$\bar{x}^\mu = (\bar{x}^0, x^i, \bar{x}^3), \quad \bar{x}^0 = x^0 + x^3, \quad \bar{x}^3 = Ax^0 + Bx^3, \quad (3)$$

where  $A, B$  are real constants [11], does allow for a description of statistical mechanics as long as  $A \pm B \neq 0$ . In this case, the temperature cannot be identified with the temperature associated with the theory in Minkowski space (quantized on an equal time surface), although they will be related by a multiplicative factor.

Since one of the goals of light-front field theories is to study nonperturbative phenomena using the simplicity of the vacuum structure, it is best carried out in the operator formalism. Similarly, questions such as the zero modes and spontaneous symmetry breaking play an important role in light-front field theories and can be systematically studied in an operatorial formalism. To carry out such studies at finite temperature one would need to use the formalism of thermo field dynamics where one defines a thermal vacuum starting from a doubled Hilbert space of the original theory through a Bogoliubov transformation [12, 13]. It is essential, therefore, that one understands the questions of operator quantization to construct the thermal vacuum in such theories. It is with this goal that we have chosen to study systematically the quantization of theories in a general light-front frame in this paper.

The paper is organized as follows. In section II, we give some details on the properties of various quantities of interest in the general light-front frame. In section III, we carry out the classical Hamiltonian analysis for a scalar field theory in such a general frame and subsequently quantize this theory. The field decomposition into positive and negative frequency parts needs to be done carefully which we discuss. We derive all the necessary relations and derive the Feynman propagator from the vacuum expectation value of the time ordered product of fields. In section IV, we introduce various properties of the Dirac matrices as well as projection operators in this general frame and carry out the Hamiltonian analysis. The Hamiltonian analysis can be carried out both in the full spinor space or in the projected subspaces and lead to the same results. We only discuss the analysis in the projected space for simplicity. To quantize such a theory, we also solve the Dirac equation in this general frame and obtain the positive and negative energy spinors. The field decomposition can then be carried out into positive and negative frequency states much

like in the scalar case. We obtain the quantization conditions and derive the Feynman propagator from the vacuum expectation value of the time ordered product of fields. In section V, we quantize the non-Abelian gauge field theory in the light-cone gauge and derive the Feynman propagator. We show that it is doubly transverse as is the case [6] in the conventional light-front quantization using (1). We conclude with a brief summary in section VI.

## II. GENERAL LIGHT-FRONT FRAME

As discussed in (3) in the introduction, we define the general light-front frame as the frame where the coordinates have the form

$$\bar{x}^0 = x^0 + x^3, \quad \bar{x}^3 = Ax^0 + Bx^3, \quad \bar{x}^i = x^i, \quad i = 1, 2, \quad (4)$$

with  $A, B$  real constants. Here  $\bar{x}^i = x^i, i = 1, 2$  are known as transverse coordinates. For  $B = -A = -1$ , we have the conventional light-front frame (1) whereas for  $A = 0, B = 1$ , we have the oblique light-front frame (2) used in the statistical description of light-front theories [8, 9, 10]. The new coordinates in (4) are related to the old Minkowski coordinates through a linear transformation,

$$\bar{x}^\mu = L^\mu_\alpha x^\alpha, \quad x^\alpha = L^\alpha_\mu \bar{x}^\mu, \quad (5)$$

where

$$L^\mu_\alpha = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ A & 0 & 0 & B \end{pmatrix}, \quad L^\alpha_\mu = \begin{pmatrix} -\frac{B}{A-B} & 0 & 0 & \frac{1}{A-B} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{A}{A-B} & 0 & 0 & -\frac{1}{A-B} \end{pmatrix}. \quad (6)$$

From definition (5), it is easy to see that

$$L^\mu_\alpha L^\alpha_\nu = \delta^\mu_\nu, \quad L^\alpha_\mu L^\mu_\beta = \delta^\alpha_\beta, \quad (7)$$

which can also be explicitly checked from the representation in (6).

Under a change of frame (5), it is clear from (7) that scalars remain invariant while vectors and tensors transform. In particular, the metric tensor transforms as

$$\begin{aligned} \bar{g}^{\mu\nu} &= L^\mu_\alpha \eta^{\alpha\beta} L^\nu_\beta = \begin{pmatrix} 0 & 0 & 0 & (A-B) \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ (A-B) & 0 & 0 & (A^2 - B^2) \end{pmatrix}, \\ \bar{g}_{\mu\nu} &= L^\alpha_\mu \eta_{\alpha\beta} L^\beta_\nu = \begin{pmatrix} -\frac{A+B}{A-B} & 0 & 0 & \frac{1}{A-B} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \frac{1}{A-B} & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (8)$$

where  $\eta^{\alpha\beta}, \eta_{\alpha\beta}$  represent the usual Minkowski space metric (with signatures  $(+, -, -, -)$ ). Since the transformation does not necessarily represent a Lorentz transformation, the metric transforms in general. It follows from this that

$$\det(-\bar{g}_{\mu\nu}) = (-\bar{g}) = \frac{1}{(A-B)^2} > 0, \quad \sqrt{-\bar{g}} = \frac{1}{|A-B|}, \quad \sqrt{-\bar{g}}(A-B) = \text{sgn}(A-B). \quad (9)$$

We note that for  $\bar{x}^0$  to represent the time coordinate in the transformed frame, we must have [9]

$$\bar{g}_{00} \geq 0, \quad \Rightarrow \quad |B| \geq |A|. \quad (10)$$

Let us next note that a covariant vector transforms under such a change of frame as

$$\bar{V}_\mu = L^\alpha_\mu V_\alpha, \quad (11)$$

which leads to the transformation of the energy-momentum four-vector as

$$\bar{p}_0 = \frac{1}{(A-B)} (-Bp_0 + Ap_3), \quad \bar{p}_3 = \frac{1}{(A-B)} (p_0 - p_3), \quad \bar{p}_i = p_i. \quad (12)$$

The Einstein relation,

$$\bar{p}^2 = \bar{g}^{\mu\nu} \bar{p}_\mu \bar{p}_\nu = m^2,$$

in this case, leads to

$$\begin{aligned} 2(A-B)\bar{p}_0\bar{p}_3 &= \bar{p}_i^2 + (B^2 - A^2)\bar{p}_3^2 + m^2 \\ \text{or, } \bar{p}_0 &= \frac{\omega_{\bar{p}}^2}{2(A-B)\bar{p}_3}, \end{aligned} \quad (13)$$

where we have defined

$$\omega_{\bar{p}}^2 = \bar{p}_i^2 + (B^2 - A^2)\bar{p}_3^2 + m^2. \quad (14)$$

It is easily seen using (10) that this quantity is positive definite and reduces to the corresponding definition in the conventional light-front frame [3, 4] as well as the oblique light-front frame [10] for particular values of  $A, B$  noted earlier.

We note that the invariant volumes in the coordinate and momentum spaces are given by

$$\int d^4\bar{x} \sqrt{-\bar{g}} = \int d^4x, \quad \int d^4\bar{p} (\sqrt{-\bar{g}})^{-1} = \int d^4p. \quad (15)$$

Note also from (8) that

$$\bar{g}_{00} = -\frac{A+B}{A-B}, \quad (16)$$

which vanishes for the conventional light-front frame (1) making a statistical description impossible while for the oblique light-front coordinates in (2)  $\bar{g}_{00} = 1$  leading to a statistical description where the temperature can be identified with that of the original Minkowski frame [9]. We note that for any  $A, B$  such that  $A \pm B \neq 0$  (namely,  $\bar{g}_{00} \neq 0$  or divergent), a statistical description is possible with a nontrivial scaling of the temperature. In the following sections, we will quantize scalar, fermion and gauge field theories in a general light-front frame with arbitrary  $A, B$ .

Finally, for completeness, we note that under a Lorentz boost along the  $z$ -axis,

$$\tilde{x}^0 = \gamma(x^0 + \beta x^3), \quad \tilde{x}^3 = \gamma(x^3 + \beta x^0), \quad \gamma = \cosh \phi, \quad \beta\gamma = \sinh \phi,$$

so that we have

$$\tilde{\bar{x}}^0 = e^\phi \bar{x}^0, \quad \tilde{\bar{x}}^3 = e^{-\phi} \bar{x}^3 + (A+B) \sinh \phi \bar{x}^0. \quad (17)$$

We see that such a boost acts as a scale transformation for  $\bar{x}^0$ , but not for  $\bar{x}^3$  in general, unless  $A+B=0$  corresponding to the conventional light-front frame (1). However, the quantization surface  $\bar{x}^0 = 0$  remains invariant under such a transformation. Similarly, it can be seen that the generators  $J_3, E_1 = -K_1 + J_2, E_2 = -K_2 - J_1$ , where  $J_i$  and  $K_i$  correspond to angular momentum and boost operators respectively, also leave the surface of quantization invariant. Consequently,  $J_3, K_3, E_1, E_2$  correspond to kinematical generators much like in the conventional light-front frame [14]. In addition, it is clear that translations along the  $x^i, (x^0 - x^3)$  leave the quantization surface invariant leading to the fact that  $P_i, (P_0 - P_3)$  are kinematical generators as well [7].

### III. SCALAR FIELDS

In this section, we will carry out the Hamiltonian analysis for the scalar field and quantize it in the general light-front frame. Using the transformation laws discussed in the previous section, it is easily seen that the action for a free scalar field can be written as

$$S = \int d^4\bar{x} \sqrt{-\bar{g}} \mathcal{L}, \quad (18)$$

where

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} (\bar{g}^{\mu\nu} \bar{\partial}_\mu \phi \bar{\partial}_\nu \phi - m^2 \phi^2) \\ &= (A-B) \bar{\partial}_0 \phi \bar{\partial}_3 \phi - \frac{1}{2} (\bar{\partial}_i \phi)^2 - \frac{1}{2} (B^2 - A^2) (\bar{\partial}_3 \phi)^2 - \frac{m^2}{2} \phi^2. \end{aligned} \quad (19)$$

The conjugate momentum density can now be defined in the standard manner as

$$\Pi = \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial\bar{\partial}_0\phi} = \sqrt{-g}(A-B)\bar{\partial}_3\phi = \text{sgn}(A-B)\bar{\partial}_3\phi, \quad (20)$$

where we have used (9). This leads to the only primary constraint of the theory of the form [1]

$$\chi = \Pi - \text{sgn}(A-B)\bar{\partial}_3\phi \approx 0. \quad (21)$$

The Hamiltonian density together with the primary constraint takes the form

$$\mathcal{H} = \frac{\sqrt{-g}}{2} \left( (\bar{\partial}_i\phi)^2 + (B^2 - A^2)(\bar{\partial}_3\phi)^2 + m^2\phi^2 \right) + \lambda\chi, \quad (22)$$

where  $\lambda$  represents the Lagrange multiplier to be determined. Requiring the primary constraint to be stationary, we obtain

$$\begin{aligned} \dot{\chi}(\bar{x}) &= \{\chi(\bar{x}), H\} \\ &= \sqrt{-g} \left( \bar{\partial}_i^2\phi + (B^2 - A^2)\bar{\partial}_3^2\phi - m^2\phi - 2(A-B)\bar{\partial}_3\phi \right) \approx 0, \end{aligned} \quad (23)$$

where a “dot” denotes derivative with respect to  $\bar{x}^0$  and in evaluating the Poisson bracket above, we have used the canonical Poisson brackets between variables (for equal  $\bar{x}^0$  coordinates), namely,

$$\{\phi(\bar{x}), \phi(\bar{y})\} = 0 = \{\Pi(\bar{x}), \Pi(\bar{y})\}, \quad \{\phi(\bar{x}), \Pi(\bar{y})\} = \delta^3(\bar{x} - \bar{y}). \quad (24)$$

Equation (23) determines the Lagrange multiplier  $\lambda$  and shows that there are no further constraints in the theory except for the primary constraint. The equal “time” ( $\bar{x}^0$ ) Poisson bracket between the primary constraint leads to the matrix

$$\{\chi(\bar{x}), \chi(\bar{y})\} = C(\bar{x}, \bar{y}) = -2\text{sgn}(A-B)\bar{\partial}_3^{(\bar{x})}\delta^2(\bar{x}_\perp - \bar{y}_\perp)\delta(\bar{x}^3 - \bar{y}^3), \quad (25)$$

where we have identified collectively  $\bar{x}_\perp = (\bar{x}^i)$ . The inverse of this matrix of constraints is easily obtained to be

$$C^{-1}(\bar{x}, \bar{y}) = -\frac{\text{sgn}(A-B)}{2}\delta^2(\bar{x}_\perp - \bar{y}_\perp)\epsilon(\bar{x}^3 - \bar{y}^3), \quad (26)$$

and we have defined the alternating step function as

$$\epsilon(x) = \frac{1}{2}(\theta(x) - \theta(-x)), \quad (27)$$

such that

$$\partial\epsilon(x) = \delta(x). \quad (28)$$

With this the Dirac brackets between the variables can be calculated. The independent (equal time) bracket takes the form

$$\{\phi(\bar{x}), \phi(\bar{y})\}_D = -\frac{\text{sgn}(A-B)}{2}\delta^2(\bar{x}_\perp - \bar{y}_\perp)\epsilon(\bar{x}^3 - \bar{y}^3). \quad (29)$$

Since the constraints can be set strongly equal to zero in the Dirac brackets, the brackets between other variables can be easily obtained from this using the primary constraint (21). Furthermore, in going to the quantum theory, we can obtain the basic, independent (equal time) commutation relation between fields from (29) to be

$$[\phi(\bar{x}), \phi(\bar{y})] = -\frac{i\text{sgn}(A-B)}{2}\delta^2(\bar{x}_\perp - \bar{y}_\perp)\epsilon(\bar{x}^3 - \bar{y}^3), \quad (30)$$

where we have assumed  $\hbar = 1$  (which we will assume throughout this paper).

In the general light-front frame, the decomposition of the fields into positive and negative energy parts has to be done carefully and since it is essential for the subsequent discussions, we discuss this in some detail. We note that since the scalar field satisfies the equation

$$(\bar{g}^{\mu\nu}\bar{\partial}_\mu\bar{\partial}_\nu + m^2)\phi(\bar{x}) = 0,$$

the field decomposition takes the form

$$\begin{aligned}\phi(\bar{x}) &= \frac{1}{(2\pi)^{3/2}} \int d^4\bar{k} (\sqrt{-g})^{-1} \delta(\bar{k}^2 - m^2) e^{-i\bar{k}\cdot\bar{x}} \hat{\phi}(\bar{k}) \\ &= \frac{1}{(2\pi)^{3/2}} \int d^4\bar{k} (\sqrt{-g})^{-1} \delta(2(A-B)\bar{k}_0\bar{k}_3 - \omega_{\bar{k}}^2) e^{-i\bar{k}\cdot\bar{x}} \hat{\phi}(\bar{k}),\end{aligned}\quad (31)$$

where  $\omega_{\bar{k}}$  is defined in (14). We note that the delta function constraint leads to

$$\bar{k}_0 = \frac{\omega_{\bar{k}}^2}{2(A-B)\bar{k}_3}.$$

As a result, the sign of the energy depends on the sign of  $(A-B)$  and if we integrate out  $\bar{k}_0$ , we cannot obtain a clean separation into positive and negative energy terms in the usual manner. Let us, therefore, scale and define

$$(\bar{k}_i, \bar{k}_3) \rightarrow \text{sgn}(A-B) (\bar{k}_i, \bar{k}_3), \quad \tilde{\bar{k}} = (\bar{k}_0, \text{sgn}(A-B)\bar{k}_i, \text{sgn}(A-B)\bar{k}_3), \quad (32)$$

so that the field decomposition (31) can be written as

$$\begin{aligned}\phi(\bar{x}) &= \frac{1}{(2\pi)^{3/2}} \int d^2\bar{k}_\perp \int_{-\infty}^{\infty} d\bar{k}_3 \frac{(\sqrt{-g})^{-1}}{2|A-B||\bar{k}_3|} e^{-i\tilde{\bar{k}}\cdot\bar{x}} \hat{\phi}(\tilde{\bar{k}}) \\ &= \frac{1}{(2\pi)^{3/2}} \int d^2\bar{k}_\perp \int_0^{\infty} \frac{d\bar{k}_3}{2\bar{k}_3} \left( e^{-i\tilde{\bar{k}}\cdot\bar{x}} a(\bar{k}) + e^{i\tilde{\bar{k}}\cdot\bar{x}} a^\dagger(\bar{k}) \right),\end{aligned}\quad (33)$$

where we have used (9) and have identified

$$\bar{k}_0 = \bar{\omega} = \frac{\omega_{\bar{k}}^2}{2|A-B|\bar{k}_3} > 0, \quad a(\bar{k}) = \hat{\phi}(\tilde{\bar{k}}), \quad a^\dagger(\bar{k}) = \hat{\phi}(-\tilde{\bar{k}}). \quad (34)$$

With this, the decomposition of the field into positive and negative energy parts is now complete and, consequently, we can identify  $a(\bar{k})$  and  $a^\dagger(\bar{k})$  with annihilation and creation operators respectively.

Requiring the field variables to satisfy the commutation relation (30), it can be easily determined that the basic non-vanishing commutation relation of the operators  $a, a^\dagger$  takes the form

$$[a(\bar{k}), a^\dagger(\bar{k}')] = 2\bar{k}_3 \delta^3(\bar{k} - \bar{k}'). \quad (35)$$

This shows that the basic commutation relation between the operators  $a(\bar{k}), a^\dagger(\bar{k})$  remains the same in any general light-front frame (when the decomposition into positive and negative energy parts has been properly carried out) and the frame dependence is really contained in the spatial part of the plane wave solutions. We would like to emphasize here that had we not carefully carried out the decomposition into positive and negative frequency parts through the use of (32) and (34), the commutation relation (35) would involve a factor of  $\text{sgn}(A-B)$ . For bosonic theories, this is not a problem and would simply imply that depending on the sign of  $(A-B)$ , the roles of  $a, a^\dagger$  have to be interchanged. However, the problem is more serious for theories involving fermions where one cannot use an expansion involving positive and negative spinors until a careful separation into positive and negative energy parts has been carried out.

Given the basic commutation relations (35), the two point function can now be calculated easily and leads to

$$\langle 0 | \phi(\bar{x}) \phi(\bar{y}) | 0 \rangle = \frac{1}{(2\pi)^3} \int d^2\bar{k}_\perp \int_0^{\infty} \frac{d\bar{k}_3}{2\bar{k}_3} e^{-i\tilde{\bar{k}}\cdot(\bar{x}-\bar{y})}. \quad (36)$$

Using the integral representation for the step function

$$\theta(\bar{x}^0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \frac{e^{i\omega' \bar{x}^0}}{\omega' - i\epsilon}, \quad (37)$$

it can be shown with some algebra that the Feynman Green's function of the theory has the form

$$\begin{aligned}iG_F(\bar{x} - \bar{y}) &= \langle 0 | T(\phi(\bar{x})\phi(\bar{y})) | 0 \rangle = \theta(\bar{x}^0 - \bar{y}^0) \langle 0 | \phi(\bar{x})\phi(\bar{y}) | 0 \rangle + \theta(\bar{y}^0 - \bar{x}^0) \langle 0 | \phi(\bar{y})\phi(\bar{x}) | 0 \rangle \\ &= \int \frac{d^4\bar{k} (\sqrt{-g})^{-1}}{(2\pi)^4} \frac{i}{\bar{k}^2 - m^2 + i\epsilon} e^{-i\bar{k}\cdot(\bar{x}-\bar{y})}.\end{aligned}\quad (38)$$

Consequently, we can identify the momentum space Feynman propagator as

$$iG_F(\bar{k}) = \frac{i}{\bar{k}^2 - m^2 + i\epsilon} = \frac{i}{2(A - B)\bar{k}_0\bar{k}_3 - \omega_{\bar{k}}^2 + i\epsilon}. \quad (39)$$

This coincides with the conventional light-front propagator [4] when  $B = -A = -1$  as well as with the propagator in the oblique coordinates [8, 10] when  $A = 0, B = 1$ .

#### IV. FERMION FIELDS

In dealing with fermion theories, we note that the Dirac matrices,  $\gamma^\mu$ , would transform like coordinate vectors [10] so that in the new frame we have

$$\bar{\gamma}^\mu = L^\mu{}_\alpha \gamma^\alpha. \quad (40)$$

Explicitly, this leads to

$$\bar{\gamma}^0 = \gamma^0 + \gamma^3, \quad \bar{\gamma}^3 = A\gamma^0 + B\gamma^3, \quad \bar{\gamma}^i = \gamma^i. \quad (41)$$

The transformed matrices satisfy the Clifford algebra

$$\{\bar{\gamma}^\mu, \bar{\gamma}^\nu\} = 2\bar{g}^{\mu\nu}, \quad (42)$$

and from the form of the metric in (8), this leads to

$$\begin{aligned} (\bar{\gamma}^0)^2 &= 0, \\ \{\bar{\gamma}^0, \bar{\gamma}^3\} &= 2(A - B), \\ (\bar{\gamma}^3)^2 &= A^2 - B^2 < 0, \\ \{\bar{\gamma}^i, \bar{\gamma}^j\} &= 2\eta^{ij}, \quad i, j = 1, 2, \\ \{\bar{\gamma}^i, \bar{\gamma}^0\} &= 0 = \{\bar{\gamma}^i, \bar{\gamma}^3\}, \end{aligned} \quad (43)$$

where we have used (10).

Given the transformed Dirac matrices, let us define two projection operators,

$$P^+ = \frac{1}{2(A - B)} \bar{\gamma}^3 \bar{\gamma}^0, \quad P^- = \frac{1}{2(A - B)} \bar{\gamma}^0 \bar{\gamma}^3. \quad (44)$$

It is easy to check that these satisfy

$$(P^+)^2 = P^+, \quad (P^-)^2 = P^-, \quad P^\pm P^\mp = 0, \quad P^+ + P^- = \mathbf{1}. \quad (45)$$

Thus, these define orthogonal projection operators for any value of the constants  $A, B$ . The transformed Dirac matrices do not have very simple Hermiticity properties. For example,

$$\begin{aligned} (\bar{\gamma}^0)^\dagger &= \frac{1}{A - B} (-(A + B)\bar{\gamma}^0 + 2\bar{\gamma}^3), \\ (\bar{\gamma}^3)^\dagger &= \frac{1}{A - B} (-2AB\bar{\gamma}^0 + (A + B)\bar{\gamma}^3), \\ (\bar{\gamma}^i)^\dagger &= -\bar{\gamma}^i. \end{aligned} \quad (46)$$

In spite of this, the projection operators can be easily checked to be Hermitian, namely,

$$(P^+)^\dagger = P^+, \quad (P^-)^\dagger = P^-. \quad (47)$$

The projection operators can also be seen to satisfy various useful relations,

$$P^+ \bar{\gamma}^0 = 0 = \bar{\gamma}^0 P^-, \quad \bar{\gamma}^3 P^+ = P^- \bar{\gamma}^3, \quad \bar{\gamma}^0 P^+ = P^- \bar{\gamma}^0, \quad \bar{\gamma}^3 P^- = P^+ \bar{\gamma}^3, \quad \bar{\gamma}^i P^\pm = P^\pm \bar{\gamma}^i, \quad (48)$$

and so on.

For completeness, we note here that if we use the Bjorken-Drell representation [15] for the original Dirac matrices, then the transformed ones will have the explicit forms (we do not write the form of  $\bar{\gamma}^i$  which remains the same)

$$\begin{aligned}\bar{\gamma}^0 &= \begin{pmatrix} \mathbf{1} & \sigma_3 \\ -\sigma_3 & -\mathbf{1} \end{pmatrix}, \\ \bar{\gamma}^3 &= \begin{pmatrix} A\mathbf{1} & B\sigma_3 \\ -B\sigma_3 & -A\mathbf{1} \end{pmatrix}, \\ P^\pm &= \frac{1}{2} \begin{pmatrix} \mathbf{1} & \pm\sigma_3 \\ \pm\sigma_3 & \mathbf{1} \end{pmatrix}.\end{aligned}\tag{49}$$

Here each of the elements represents a  $2 \times 2$  matrix. The important thing to note here is that the projection operators,  $P^\pm$ , are independent of the values of the constants  $A, B$ .

With these basics, we note that the action for a free, massive fermion can be written as

$$\begin{aligned}S &= \int d^4x \mathcal{L} = \int d^4x \bar{\psi} (i\gamma^\alpha \partial_\alpha - m) \psi \\ &= \int d^4\bar{x} \sqrt{-\bar{g}} \psi^\dagger \left[ 2iP^+ \bar{\partial}_0 + i((A+B)P^+ + (A-B)P^-) \bar{\partial}_3 + i\alpha_i \bar{\partial}_i - \frac{m}{A-B} (-B\bar{\gamma}^0 + \bar{\gamma}^3) \right] \psi,\end{aligned}\tag{50}$$

where we have defined as usual

$$\alpha_i = \gamma^0 \gamma^i.\tag{51}$$

Let us define the projected fermions

$$\psi_\pm = P^\pm \psi.\tag{52}$$

The Hamiltonian analysis for the Dirac theory can be carried out either in terms of the projected spinor fields or in terms of the original spinor field and they lead to the same result. (We have carried out both of these analyses.) However, for simplicity, we will only describe here the Hamiltonian analysis in terms of the projected spinor fields.

In terms of the projected spinor fields, the action for the free fermion field (50) can be written as

$$\begin{aligned}S &= \int d^4\bar{x} \sqrt{-\bar{g}} \left[ \psi_+^\dagger (2i\bar{\partial}_0 + i(A+B)\bar{\partial}_3) \psi_+ + i(A-B) \psi_-^\dagger \bar{\partial}_3 \psi_- \right. \\ &\quad \left. + i\psi_+^\dagger \alpha_i \bar{\partial}_i \psi_- + i\psi_-^\dagger \alpha_i \bar{\partial}_i \psi_+ - \frac{m}{A-B} \psi_+^\dagger \bar{\gamma}^3 \psi_- - \frac{m}{2} \psi_-^\dagger \bar{\gamma}^0 \psi_+ \right].\end{aligned}\tag{53}$$

Using (46) it can be checked that the action is Hermitian.

The canonical momentum densities can now be determined from the action and lead to (we use a left derivative for the fermions)

$$\begin{aligned}\Pi_+^\dagger &= \frac{\partial(\sqrt{-\bar{g}}\mathcal{L})}{\partial\bar{\partial}_0\psi_+} = -2i\sqrt{-\bar{g}}\psi_+^\dagger, \\ \Pi_+ &= \frac{\partial(\sqrt{-\bar{g}}\mathcal{L})}{\partial\bar{\partial}_0\psi_+^\dagger} = 0, \\ \Pi_-^\dagger &= \frac{\partial(\sqrt{-\bar{g}}\mathcal{L})}{\partial\bar{\partial}_0\psi_-} = 0, \\ \Pi_- &= \frac{\partial(\sqrt{-\bar{g}}\mathcal{L})}{\partial\bar{\partial}_0\psi_-^\dagger} = 0,\end{aligned}\tag{54}$$

where we have suppressed the spinor indices for simplicity. We note that there are four primary constraints in the theory, namely,

$$\chi_1^\dagger = \Pi_+^\dagger + 2i\sqrt{-\bar{g}}\psi_+^\dagger \approx 0, \quad \chi_2 = \Pi_+ \approx 0, \quad \chi_3^\dagger = \Pi_-^\dagger \approx 0, \quad \chi_4 = \Pi_- \approx 0.\tag{55}$$

Consequently, adding the primary constraints, we can write the Hamiltonian density for the theory to be

$$\begin{aligned} \mathcal{H} = & \sqrt{-g} \left[ -i(A+B)\psi_+^\dagger \bar{\partial}_3 \psi_+ - i(A-B)\psi_-^\dagger \bar{\partial}_3 \psi_- - i\psi_+^\dagger \alpha_i \bar{\partial}_i \psi_- - i\psi_-^\dagger \alpha_i \bar{\partial}_i \psi_+ \right. \\ & \left. + \frac{m}{A-B} \psi_+^\dagger \bar{\gamma}^3 \psi_- + \frac{m}{2} \psi_-^\dagger \bar{\gamma}^0 \psi_+ \right] + \chi_1^\dagger \lambda_1 + \lambda_2^\dagger \chi_2 + \chi_3^\dagger \lambda_3 + \lambda_4^\dagger \chi_4. \end{aligned} \quad (56)$$

Here  $\lambda_1, \lambda_2^\dagger, \lambda_3, \lambda_4^\dagger$  represent the Lagrange multipliers with obvious projections.

The Hamiltonian analysis can now be carried out using the canonical equal time (equal  $\bar{x}^0 = \bar{y}^0$ ) Poisson brackets

$$\left\{ \psi_{\pm,a}(\bar{x}), \Pi_{\pm,b}^\dagger(\bar{y}) \right\} = - (P^\pm)_{ab} \delta^3(\bar{x} - \bar{y}), \quad \left\{ \psi_{\pm,a}^\dagger(\bar{x}), \Pi_{\pm,b}(\bar{y}) \right\} = - (P^\pm)_{ba} \delta^3(\bar{x} - \bar{y}), \quad a, b = 1, 2, 3, 4, \quad (57)$$

with all others vanishing. Requiring the primary constraints (55) to be stationary determines the Lagrange multipliers  $\lambda_1, \lambda_2^\dagger$  and leads to two secondary constraints

$$\chi_5^\dagger = i(A-B)\bar{\partial}_3 \psi_-^\dagger + i\bar{\partial}_i \psi_+^\dagger \alpha_i + \frac{m}{A-B} \psi_+^\dagger \bar{\gamma}^3 \approx 0, \quad \chi_6 = i(A-B)\bar{\partial}_3 \psi_- + i\alpha_i \bar{\partial}_i \psi_+ - \frac{m}{2} \bar{\gamma}^0 \psi_+ \approx 0. \quad (58)$$

Requiring these to be stationary determines the remaining two Lagrange multipliers  $\lambda_3, \lambda_4^\dagger$  and the chain of constraints terminates. Thus, there are six constraints (55) and (58) and it can be easily verified that they are all second class. The Dirac brackets can now be obtained iteratively in a systematic manner and we note the final form of the nontrivial (equal time) Dirac brackets involving the field variables,

$$\begin{aligned} \left\{ \psi_{+,a}(\bar{x}), \psi_{+,b}^\dagger(\bar{y}) \right\}_D &= -\frac{i}{2\sqrt{-g}} (P^+)_{ab} \delta^3(\bar{x} - \bar{y}), \\ \left\{ \psi_{-,a}(\bar{x}), \psi_{+,b}^\dagger(\bar{y}) \right\}_D &= \frac{\text{sgn}(A-B)}{2} \left( P^- \left( i\alpha_i \bar{\partial}_i - \frac{m}{2} \bar{\gamma}^0 \right) P^+ \right)_{ab} \delta^2(\bar{x}_\perp - \bar{y}_\perp) \epsilon(\bar{x}^3 - \bar{y}^3), \\ \left\{ \psi_{+,a}(\bar{x}), \psi_{-,b}^\dagger(\bar{y}) \right\}_D &= \frac{\text{sgn}(A-B)}{2} \left( P^+ \left( i\alpha_i \bar{\partial}_i - \frac{m}{A-B} \bar{\gamma}^3 \right) P^- \right)_{ab} \delta^2(\bar{x}_\perp - \bar{y}_\perp) \epsilon(\bar{x}^3 - \bar{y}^3), \\ \left\{ \psi_{-,a}(\bar{x}), \psi_{-,b}^\dagger(\bar{y}) \right\}_D &= \frac{i}{2|A-B|} \left( P^- \left( i\alpha_i \bar{\partial}_i - \frac{m}{2} \bar{\gamma}^0 \right) \left( i\alpha_j \bar{\partial}_j - \frac{m}{A-B} \bar{\gamma}^3 \right) P^- \right)_{ab} \delta^2(\bar{x}_\perp - \bar{y}_\perp) \\ &\quad \times \int d\bar{z}^3 \epsilon(\bar{x}^3 - \bar{z}^3) \epsilon(\bar{z}^3 - \bar{y}^3). \end{aligned} \quad (59)$$

For  $B = -A = -1$ , these can be seen to coincide with the Dirac brackets [16] derived in the conventional light-front frame (1). Since the constraints can now be set equal to zero strongly and  $\psi_-$  is a constrained field (see (58)), only the first of these relations is independent. Every other bracket can be derived from this using the constraint relations. In going over to the quantum theory, we can take over the Dirac brackets to anti-commutation relations and the nontrivial equal time anti-commutation relation takes the form

$$\left[ \psi_{+,a}(\bar{x}), \psi_{+,b}^\dagger(\bar{y}) \right]_+ = \frac{1}{2\sqrt{-g}} (P^+)_{ab} \delta^3(\bar{x} - \bar{y}). \quad (60)$$

The spinor solutions for the theory can be worked out in the projected space quite easily (we have worked these out in the full theory as well and they are completely equivalent). We note that in the plane wave basis, the Dirac equation takes the form

$$\begin{aligned} (2\bar{k}_0 + (A+B)\bar{k}_3) u_+(\bar{k}) + \left( \alpha_i \bar{k}_i - \frac{m}{A-B} \bar{\gamma}^3 \right) u_-(\bar{k}) &= 0, \\ (A-B)\bar{k}_3 u_-(\bar{k}) + \left( \alpha_i \bar{k}_i - \frac{m}{2} \bar{\gamma}^0 \right) u_+(\bar{k}) &= 0. \end{aligned} \quad (61)$$

The spinors can be easily checked to satisfy the Einstein relation (13). Here by definition, the projected spinors  $u_\pm = P^\pm u_\pm$  have to have the forms

$$u_+ = \begin{pmatrix} u_1 \\ \sigma_3 u_1 \end{pmatrix}, \quad u_- = \begin{pmatrix} u_2 \\ -\sigma_3 u_2 \end{pmatrix}, \quad (62)$$



which can be seen from the definitions in (49). Here  $u_1, u_2$  are two component spinors. Substituting these into (61), it can be determined that, for positive energy,

$$u_+(\vec{k}) = \sqrt{\frac{(A-B)\bar{k}_3}{2}} \begin{pmatrix} u_1 \\ \sigma_3 u_1 \end{pmatrix}, \quad u_-(\vec{k}) = \frac{1}{\sqrt{2(A-B)\bar{k}_3}} \begin{pmatrix} (m - \sigma_i \bar{k}_i \sigma_3) u_1 \\ -\sigma_3 (m - \sigma_i \bar{k}_i \sigma_3) u_1 \end{pmatrix}. \quad (63)$$

The spinors are normalized such that the positive energy spinor  $u = u_+ + u_-$  satisfies the conventional normalization

$$\bar{u}(\vec{k})u(\vec{k}) = 2m. \quad (64)$$

We note that with the choices

$$u_1^{(\uparrow)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_1^{(\downarrow)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (65)$$

we can obtain the spin-up and spin-down spinor states. Thus, for example, we have

$$\begin{aligned} u^{(\uparrow)} &= \frac{1}{\sqrt{2(A-B)\bar{k}_3}} \begin{pmatrix} (A-B)\bar{k}_3 + m \\ -(\bar{k}_1 + i\bar{k}_2) \\ (A-B)\bar{k}_3 - m \\ -(\bar{k}_1 + i\bar{k}_2) \end{pmatrix}, \quad u_+^{(\uparrow)} = \sqrt{\frac{(A-B)\bar{k}_3}{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \\ u^{(\downarrow)} &= \frac{1}{\sqrt{2(A-B)\bar{k}_3}} \begin{pmatrix} \bar{k}_1 - i\bar{k}_2 \\ (A-B)\bar{k}_3 + m \\ -(\bar{k}_1 - i\bar{k}_2) \\ -(A-B)\bar{k}_3 + m \end{pmatrix}, \quad u_+^{(\downarrow)} = \sqrt{\frac{(A-B)\bar{k}_3}{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}. \end{aligned} \quad (66)$$

The charge conjugate spinors are obtained from the defining relation

$$v = C\bar{u}^T = i\gamma^2 u^*, \quad (67)$$

which leads, for example, to

$$\begin{aligned} v^{(\uparrow)} &= \frac{1}{\sqrt{2(A-B)\bar{k}_3}} \begin{pmatrix} -(\bar{k}_1 - i\bar{k}_2) \\ -(A-B)\bar{k}_3 + m \\ \bar{k}_1 - i\bar{k}_2 \\ (A-B)\bar{k}_3 + m \end{pmatrix}, \quad v_+^{(\uparrow)} = \sqrt{\frac{(A-B)\bar{k}_3}{2}} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \\ v^{(\downarrow)} &= \frac{1}{\sqrt{2(A-B)\bar{k}_3}} \begin{pmatrix} -(A-B)\bar{k}_3 + m \\ \bar{k}_1 + i\bar{k}_2 \\ -(A-B)\bar{k}_3 - m \\ \bar{k}_1 + i\bar{k}_2 \end{pmatrix}, \quad v_+^{(\downarrow)} = \sqrt{\frac{(A-B)\bar{k}_3}{2}} \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \end{aligned} \quad (68)$$

It is easy to check that when  $B = -A = -1$ , these spinors coincide with the ones in the conventional light-front frame [14]. From their forms, it is also easily verified that

$$\sum_s u_{+,a}(\vec{k}, s) u_{+,b}^\dagger(\vec{k}, s) = (A-B)\bar{k}_3 P_{ab}^+ = \sum_s v_{+,a}(\vec{k}, s) v_{+,b}^\dagger(\vec{k}, s), \quad (69)$$

which will be useful later.

The field decomposition for the independent field component can now be carried out (much like in the scalar case) as

$$\psi_{+,a}(\bar{x}) = \frac{1}{(2\pi)^{3/2}} \sum_s \int d^2\bar{k}_\perp \int_0^\infty \frac{d\bar{k}_3}{2\bar{k}_3} \left[ e^{-i\vec{k}\cdot\bar{x}} b(\vec{k}, s) u_{+,a}(\vec{k}, s) + e^{i\vec{k}\cdot\bar{x}} d^\dagger(\vec{k}, s) v_{+,a}(\vec{k}, s) \right], \quad (70)$$

where we have used the definitions in (32) and (34) and note in particular that

$$\bar{k}_0 = \bar{\omega} = \frac{\omega_k^2}{2|A-B|\bar{k}_3} > 0.$$

This, therefore, truly leads to a separation of the fields into positive and negative frequency terms which is quite crucial in the use of the positive and negative energy spinors. Requiring the fields to satisfy the anti-commutation relation (60), it can be determined with the use of (69) that the nontrivial anti-commutation relation satisfied by the operators  $b, d$  has the form

$$[b(\bar{k}, s), b^\dagger(\bar{k}', s')]_+ = 2\bar{k}_3 \delta_{ss'} \delta^3(\bar{k} - \bar{k}') = [d(\bar{k}, s), d^\dagger(\bar{k}', s')]_+. \quad (71)$$

Once again, we see that the basic anti-commutation relations of the field variables is frame independent and the entire frame dependence is contained in the plane wave and the spinor solutions. It follows now from a direct calculation that

$$\begin{aligned} \langle 0 | \psi_{+,a}(\bar{x}) \psi_{+,b}^\dagger(\bar{y}) | 0 \rangle &= \frac{|A-B|P_{ab}^+}{2(2\pi)^3} \int d^2\bar{k}_\perp \int_0^\infty d\bar{k}_3 e^{-i\bar{k} \cdot (\bar{x} - \bar{y})}, \\ \langle 0 | \psi_{+,b}^\dagger(\bar{y}) \psi_{+,a}(\bar{x}) | 0 \rangle &= \frac{|A-B|P_{ab}^+}{2(2\pi)^3} \int d^2\bar{k}_\perp \int_0^\infty d\bar{k}_3 e^{i\bar{k} \cdot (\bar{x} - \bar{y})}, \end{aligned} \quad (72)$$

where we have used (69). The fermion propagator for the “+” component can now be determined to be

$$\begin{aligned} iS_{F,ab}^{(++)}(\bar{x} - \bar{y}) &= \langle 0 | T \left( \psi_{+,a}(\bar{x}) \psi_{+,b}^\dagger(\bar{y}) \right) | 0 \rangle = \theta(\bar{x}^0 - \bar{y}^0) \langle 0 | \psi_{+,a}(\bar{x}) \psi_{+,b}^\dagger(\bar{y}) | 0 \rangle - \theta(\bar{y}^0 - \bar{x}^0) \langle 0 | \psi_{+,b}^\dagger(\bar{y}) \psi_{+,a}(\bar{x}) | 0 \rangle \\ &= \int \frac{d^4\bar{k} (\sqrt{-\bar{g}})^{-1}}{(2\pi)^4} \frac{i(A-B)\bar{k}_3 P_{ab}^+}{\bar{k}^2 - m^2 + i\epsilon} e^{-i\bar{k} \cdot (\bar{x} - \bar{y})}, \end{aligned} \quad (73)$$

where we have used the definition of the step function in (37).

The other components of the propagator can be obtained now using the constraint relations. In the momentum space, all the components of the propagator can be written in the matrix form as

$$\begin{aligned} \begin{pmatrix} iS_F^{(++)}(\bar{k}) & iS_F^{(+-)}(\bar{k}) \\ iS_F^{(-+)}(\bar{k}) & iS_F^{(--)}(\bar{k}) \end{pmatrix} &= \begin{pmatrix} P^+ & 0 \\ 0 & P^- \end{pmatrix} \begin{pmatrix} (A-B)\bar{k}_3 & (-\alpha_i \bar{k}_i + \frac{m}{A-B} \bar{\gamma}^3) \\ (-\alpha_i \bar{k}_i + \frac{m}{2} \bar{\gamma}^0) & (2\bar{k}_0 + (A+B)\bar{k}_3) \end{pmatrix} \begin{pmatrix} P^+ & 0 \\ 0 & P^- \end{pmatrix} \\ &\times \frac{i}{\bar{k}^2 - m^2 + i\epsilon}. \end{aligned} \quad (74)$$

For  $A = 0, B = 1$ , we note that this coincides with the propagators derived in [10] (where the projection operators were not present because of the space in which the propagators were defined).

## V. GAUGE FIELD THEORY

In this section, we will quantize the Yang-Mills theory with the gauge fields belonging to  $SU(N)$  group in the conventional light-cone gauge in order to see if a doubly transverse gauge propagator [6] results in the general frame as well. The action for the theory in the general frame has the form

$$S = \int d^4\bar{x} \sqrt{-\bar{g}} \mathcal{L}, \quad (75)$$

where the Lagrangian density is given by

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} \bar{g}^{\mu\lambda} \bar{g}^{\nu\rho} \bar{F}_{\mu\nu}^\alpha \bar{F}_{\lambda\rho}^\alpha \\ &= \frac{1}{2} (A-B)^2 \bar{F}_{03}^\alpha \bar{F}_{03}^\alpha + (A-B) \bar{F}_{0i}^\alpha \bar{F}_{3i}^\alpha + \frac{1}{2} (A^2 - B^2) \bar{F}_{3i}^\alpha \bar{F}_{3i}^\alpha - \frac{1}{4} \bar{F}_{ij}^\alpha \bar{F}_{ij}^\alpha. \end{aligned} \quad (76)$$

Here,  $\alpha = 1, 2, \dots, N^2 - 1$  and the field strength tensors is defined as

$$\bar{F}_{\mu\nu}^\alpha = \bar{\partial}_\mu \bar{A}_\nu^\alpha - \bar{\partial}_\nu \bar{A}_\mu^\alpha + g f^{\alpha\beta\gamma} \bar{A}_\mu^\beta \bar{A}_\nu^\gamma, \quad (77)$$

with  $f^{\alpha\beta\gamma}$  representing the structure constants of the group. The Abelian limit can be obtained simply by setting the coupling constant  $g$  to zero. We will omit the “bars” on the field variables in the following for simplicity.

The conjugate momenta are now obtained to be

$$\begin{aligned} \Pi^{0\alpha} &= \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial\bar{\partial}_0 A_0^\alpha} = 0, \\ \Pi^{3\alpha} &= \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial\bar{\partial}_0 A_3^\alpha} = \sqrt{-g}(A-B)^2 F_{03}^\alpha, \\ \Pi^{i\alpha} &= \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial\bar{\partial}_0 A_i^\alpha} = \text{sgn}(A-B) F_{3i}^\alpha. \end{aligned} \quad (78)$$

Thus, we see that the theory has two primary constraints,

$$\chi_1^\alpha = \Pi^{0\alpha} \approx 0, \quad \chi_{2i}^\alpha = \Pi^{i\alpha} - \text{sgn}(A-B) F_{3i}^\alpha \approx 0. \quad (79)$$

Adding these primary constraints, the starting Hamiltonian density for the Hamiltonian analysis takes the form

$$\begin{aligned} \mathcal{H} &= \frac{1}{2\sqrt{-g}(A-B)^2} \Pi^{3\alpha} \Pi^{3\alpha} - A_0^\alpha (\bar{D}_i \Pi^{i\alpha} + \bar{D}_3 \Pi^{3\alpha}) - \frac{\sqrt{-g}(A^2 - B^2)}{2} F_{3i}^\alpha F_{3i}^\alpha + \frac{\sqrt{-g}}{4} F_{ij}^\alpha F_{ij}^\alpha \\ &\quad + \lambda_1^\alpha \chi_1^\alpha + \lambda_{2i}^\alpha \chi_{2i}^\alpha, \end{aligned} \quad (80)$$

where  $\lambda_1^\alpha, \lambda_{2i}^\alpha$  are Lagrange multipliers and the covariant derivative is defined as

$$\bar{D}_\mu \Pi^{\nu\alpha} = \bar{\partial}_\mu \Pi^{\nu\alpha} + g f^{\alpha\beta\gamma} A_\mu^\beta \Pi^{\nu\gamma}. \quad (81)$$

In writing the Hamiltonian density in this form, we have discarded a total divergence term. We can now use the equal time ( $\bar{x}^0$ ) canonical Poisson brackets

$$\{A_\mu^\alpha(\bar{x}), \Pi^{\nu\beta}(\bar{y})\} = \delta^{\alpha\beta} \delta_\mu^\nu \delta^3(\bar{x} - \bar{y}), \quad (82)$$

with all others vanishing. Requiring the primary constraints to be stationary, determines the Lagrange multiplier  $\lambda_{2i}^\alpha$  and leads to the secondary constraint

$$\chi_3^\alpha = \bar{D}_i \Pi^{i\alpha} + \bar{D}_3 \Pi^{3\alpha} \approx 0. \quad (83)$$

Furthermore, requiring the secondary constraint (83) to be stationary, determines the Lagrange multiplier  $\lambda_1^\alpha$  and the chain of constraints terminates.

The complete set of constraints for the theory are given by (79) and (83). It can be easily checked that of these  $\chi_1^\alpha, \chi_3^\alpha$  correspond to first class constraints while  $\chi_{2i}^\alpha$  represents a second class constraint. This is consistent with the general characteristics of a light-front theory, namely, it develops a genuine second class constraint in addition to the ones already present in the conventional theory. Since there are two first class constraints, we choose two gauge fixing conditions that will make these second class. Keeping the physical light-cone gauge in mind, we choose the gauge fixing conditions to correspond to

$$\phi_1^\alpha = \bar{n} \cdot A^\alpha \approx 0, \quad \phi_2^\alpha = \bar{\partial} \cdot A^\alpha \approx 0, \quad \bar{n}^\mu = (0, 0, 0, 1), \quad \bar{n}^2 = 0. \quad (84)$$

The Dirac brackets can now be determined iteratively and we simply note the final result for the equal time Dirac brackets involving the field variables,

$$\{A_\mu^\alpha(\bar{x}), A_\nu^\beta(\bar{y})\}_D = \frac{\delta^{\alpha\beta} \text{sgn}(A-B)}{2} P_{\mu\nu}^T(\bar{n}, \bar{\partial}) \bar{\partial}_3^{-1} \delta^3(\bar{x} - \bar{y}), \quad (85)$$

where we have defined (all the derivatives are with respect to the argument  $\bar{x}$ )

$$\bar{\partial}_3^{-1} \delta^3(\bar{x} - \bar{y}) = \delta^2(\bar{x}_\perp - \bar{y}_\perp) \epsilon(\bar{x}^3 - \bar{y}^3), \quad (86)$$

and  $P_{\mu\nu}^T(\bar{n}, \bar{\partial})$  represents the projection operator transverse to both  $\bar{n}_\mu, \bar{\partial}_\mu$  and has the form

$$P_{\mu\nu}^T(\bar{n}, \bar{\partial}) = \bar{g}_{\mu\nu} - \frac{\bar{n}_\mu \bar{\partial}_\nu + \bar{n}_\nu \bar{\partial}_\mu}{\bar{n} \cdot \bar{\partial}} + \frac{\bar{\partial}^2}{(\bar{n} \cdot \bar{\partial})^2} \bar{n}_\mu \bar{n}_\nu. \quad (87)$$

The projection operator is symmetric, homogeneous in both  $\bar{n}_\mu, \bar{\partial}_\mu$  and it is straightforward to check that it satisfies

$$\bar{n}^\mu P_{\mu\nu}^T(\bar{n}, \bar{\partial}) = 0 = \bar{\partial}^\mu P_{\mu\nu}^T(\bar{n}, \bar{\partial}). \quad (88)$$

The other equal time brackets can be obtained from this since the constraints can be strongly set equal to zero in the Dirac brackets. This also leads to the fact that in the quantum theory, we can take the independent commutation relation involving fields to be

$$[A_\mu^\alpha(\bar{x}), A_\nu^\beta(\bar{y})] = \frac{i\delta^{\alpha\beta} \text{sgn}(A-B)}{2} P_{\mu\nu}^T(\bar{n}, \bar{\partial}) \delta^2(\bar{x}_\perp - \bar{y}_\perp) \epsilon(\bar{x}^3 - \bar{y}^3). \quad (89)$$

The field decomposition can now be carried out in the standard manner as in the case of the scalar field and takes the form

$$A_\mu^\alpha(\bar{x}) = \frac{1}{(2\pi)^{3/2}} \sum_{h=1}^2 \int d^2\bar{k}_\perp \int_0^\infty \frac{d\bar{k}_3}{2\bar{k}_3} \epsilon_\mu(\bar{k}, h) \left[ e^{-i\bar{k} \cdot \bar{x}} a^\alpha(\bar{k}, h) + e^{i\bar{k} \cdot \bar{x}} a^{\dagger\alpha}(\bar{k}, h) \right], \quad (90)$$

where, as before, we have identified

$$\bar{k}_0 = \bar{\omega} = \frac{\omega_{\bar{k}}^2}{2|A-B|\bar{k}_3} > 0, \quad (91)$$

and have used (32). The two physical polarization vectors are chosen to be real for simplicity and are supposed to satisfy

$$\bar{n} \cdot \epsilon(\bar{k}, h) = 0 = \bar{k} \cdot \epsilon(\bar{k}, h), \quad \epsilon(\bar{k}, h) \cdot \epsilon(\bar{k}, h') = -\delta_{hh'}, \quad h = 1, 2. \quad (92)$$

We can choose, for example,

$$\epsilon_\mu(\bar{k}, 1) = \frac{1}{(A-B)\bar{k}_3} (\bar{k}_1, (A-B)\bar{k}_3, 0, 0), \quad \epsilon_\mu(\bar{k}, 2) = \frac{1}{(A-B)\bar{k}_3} (\bar{k}_2, 0, (A-B)\bar{k}_3, 0), \quad (93)$$

which satisfy the properties in (92). In addition to these two physical polarization vectors, we can also choose two other vectors

$$\epsilon_\mu(\bar{k}, 3) = \bar{k}_\mu, \quad \epsilon_\mu(\bar{k}, 4) = \bar{n}_\mu, \quad (94)$$

such that together they define a set of basis vectors. From the completeness of these vectors, it is easy to show that

$$\sum_{h=1}^2 \frac{\epsilon_\mu(\bar{k}, h) \epsilon_\nu(\bar{k}, h)}{\epsilon(\bar{k}, h) \cdot \epsilon(\bar{k}, h)} + \frac{\epsilon_\mu(\bar{k}, 3) \epsilon_\nu(\bar{k}, 4) + \epsilon_\nu(\bar{k}, 3) \epsilon_\mu(\bar{k}, 4)}{(\epsilon(\bar{k}, 3) \cdot \epsilon(\bar{k}, 4))} - \frac{\epsilon(\bar{k}, 3) \cdot \epsilon(\bar{k}, 4)}{(\epsilon(\bar{k}, 3) \cdot \epsilon(\bar{k}, 4))^2} \epsilon_\mu(\bar{k}, 4) \epsilon_\nu(\bar{k}, 4) = \bar{g}_{\mu\nu}, \quad (95)$$

which leads to

$$\sum_{h=1}^2 \epsilon_\mu(\bar{k}, h) \epsilon_\nu(\bar{k}, h) = -P_{\mu\nu}^T(\bar{n}, \bar{k}) = -\bar{g}_{\mu\nu} + \frac{\bar{n}_\mu \bar{k}_\nu + \bar{n}_\nu \bar{k}_\mu}{(\bar{n} \cdot \bar{k})} - \frac{\bar{k}^2}{(\bar{n} \cdot \bar{k})^2} \bar{n}_\mu \bar{n}_\nu. \quad (96)$$

This can also be constructed directly from the outer product of the forms of the polarization vectors given in (93).

Requiring the fields (90) to satisfy the commutation relations in (89), we can determine that the nontrivial commutation relation involving the operators  $a^\alpha, a^{\dagger\alpha}$  has the form

$$[a^\alpha(\bar{k}, h), a^{\dagger\beta}(\bar{k}', h')] = 2\bar{k}_3 \delta^{\alpha\beta} \delta_{hh'} \delta^3(\bar{k} - \bar{k}'). \quad (97)$$

The Feynman propagator for the theory is now straightforward to calculate. We note that

$$\langle 0 | A_\mu^\alpha(\bar{x}) A_\nu^\beta(\bar{y}) | 0 \rangle = -\frac{\delta^{\alpha\beta}}{(2\pi)^3} \int d^2\bar{k}_\perp \int_0^\infty \frac{d\bar{k}_3}{2\bar{k}_3} P_{\mu\nu}^T(\bar{n}, \bar{k}) e^{-i\bar{k} \cdot (\bar{x} - \bar{y})}. \quad (98)$$

With this we obtain,

$$\begin{aligned} iG_{F,\mu\nu}^{\alpha\beta}(\bar{x} - \bar{y}) &= \langle 0|T(A_\mu^\alpha(\bar{x})A_\nu^\beta(\bar{y}))|0\rangle = \theta(\bar{x}^0 - \bar{y}^0)\langle 0|A_\mu^\alpha(\bar{x})A_\nu^\beta(\bar{y})|0\rangle + \theta(\bar{y}^0 - \bar{x}^0)\langle 0|A_\nu^\beta(\bar{y})A_\mu^\alpha(\bar{x})|0\rangle \\ &= \int \frac{d^4\bar{k}(\sqrt{-\bar{g}})^{-1}}{(2\pi)^4} \frac{-i\delta^{\alpha\beta}P_{\mu\nu}^T(\bar{n}, \bar{k})}{\bar{k}^2 + i\epsilon} e^{-i\bar{k}\cdot(\bar{x}-\bar{y})}. \end{aligned} \quad (99)$$

It follows, therefore, that in momentum space, the Feynman propagator takes the form

$$iG_{F,\mu\nu}^{\alpha\beta}(\bar{k}) = -\left(\bar{g}_{\mu\nu} - \frac{\bar{n}_\mu\bar{k}_\nu + \bar{n}_\nu\bar{k}_\mu}{(\bar{n} \cdot \bar{k})} + \frac{\bar{k}^2}{(\bar{n} \cdot \bar{k})^2} \bar{n}_\mu\bar{n}_\nu\right) \frac{i\delta^{\alpha\beta}}{\bar{k}^2 + i\epsilon}. \quad (100)$$

We note that for any value of  $A, B$ , this propagator is doubly transverse, namely,

$$\bar{n}^\mu G_{F,\mu\nu}^{\alpha\beta}(\bar{k}) = 0 = \bar{k}^\mu G_{F,\mu\nu}^{\alpha\beta}(\bar{k}). \quad (101)$$

This has been observed earlier [6] in the conventional light-front quantization. However, our analysis shows that this is a generic feature in the general light-front frame and is a consequence of the particular choice of gauge fixing in (84).

We note here that the propagator (100) has poles at  $\bar{n} \cdot \bar{k} = 0$  in addition to the usual pole at  $\bar{k}^2 = 0$ . This necessitates a prescription for handling such poles which in the conventional equal time theories is given by the Leibbrandt-Mandelstam prescription [17]

$$\frac{1}{n \cdot k} \rightarrow \lim_{\epsilon \rightarrow 0^+} \frac{1}{n \cdot k + i\epsilon(\tilde{n} \cdot k)}, \quad (102)$$

where  $\tilde{n}^\mu$  is a dual light-like vector with  $n \cdot \tilde{n} \neq 0$ . Since the Leibbrandt-Mandelstam prescription involves only scalar combinations and as we have already argued scalar quantities do not change under a change of frame, this prescription can be readily extended to the general light-front frame. For our choice of the light-like vector in (84), we have

$$\tilde{n}^\mu = (2, 0, 0, A + B). \quad (103)$$

In our analysis, we have chosen a physical gauge fixing condition since we are interested in the Hamiltonian quantization of the theory with physical degrees of freedom. However, other gauge fixing conditions may be more useful from the path integral point of view that we have not pursued here. Some of these have been discussed in [10] within the path integral approach.

## VI. CONCLUSION

In this paper we have studied systematically the quantization of quantum field theories in a general light-front frame. We have carried out the Hamiltonian analysis for scalar, fermion as well as gauge theories. The decomposition of the fields into positive and negative frequency parts is done carefully which leads to frame independent (anti) commutation relations for the annihilation and creation operators. In the case of scalar fields and the gauge fields, the frame dependence is contained completely in the plane wave functions (as well as in the polarization vectors), while in the case of the Dirac field, the spinor solutions are frame dependent as well and we have derived these explicitly. The propagators for the various fields have been obtained from the vacuum expectation values of the time ordered products and they coincide, for specific values of  $A, B$ , with the earlier known results. In particular, we have shown that in the light-cone gauge, the gauge propagator is doubly transverse in any general frame much like it was observed to be in the conventional light-front frame [6].

We conclude by saying that all of our analysis has been carried out with a view to constructing a thermal Hilbert space within the formalism of thermofield dynamics which will then allow us to study various operatorial questions, discussed in the introduction, at finite temperature. We hope to be able to report on such a finite temperature analysis in a future publication.

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